Petrov type I Condition and Dual Fluid Dynamics

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Abstract

Recently Lysov and Strominger [arXiv:1104.5502] showed that imposing a Petrov type I condition on a $(p+1)$-dimensional timelike hypersurface embedded in a $(p+2)$-dimensional vacuum Einstein gravity reduces the degrees of freedom in the extrinsic curvature of the hypersurface to that of a fluid on the hypersurface, and that the leading-order Einstein constraint equations in terms of the mean curvature of the embedding give the non-linear incompressible Navier-Stokes equations of the dual fluid. In this paper we show that the non-relativistic fluid dual to vacuum Einstein gravity does not satisfy the Petrov type I condition at next order, unless additional constraint such as the irrotational condition is added. In addition, we show that this procedure can be inversed to derive the non-relativistic hydrodynamics with higher order corrections through imposing the Petrov type I condition, and that some second order transport coefficients can be extracted, but the dual "Petrov type I fluid" does not match the dual fluid constructed from the geometry in non-relativistic limit. We discuss the procedure both on the finite cutoff surface via the non-relativistic hydrodynamic expansion and on the highly accelerated surface via the near horizon expansion.

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1 Introduction

In the non-relativistic hydrodynamic limit, a correspondence between the the nonlinear solutions of the Einstein equations and incompressible Navier-Stokes equations is constructed in [1, 2, 3] where an intrinsically flat finite cutoff surface and regularity on the future horizon are imposed. Two equivalent presentations of the non-linear perturbed gravity solution and dual fluid expansion are given, one is for the dual fluid living on a finite cutoff surface via non-relativistic hydrodynamic expansion, the other is on the highly accelerated surface via near horizon expansion. This relation is further shown to be universal for the gravity with sphere horizon [4, 5] and with higher curvature corrections [6, 7, 8, 9, 10]. And the dual incompressible Navier-Stokes equations are found to be corrected at leading order when a non-trivial gravitational Chern-Simons term appears in the bulk [11]. More generally, the gravity is related with a fluid without gravity in one lower dimension, and related works can also be found in [12, 13, 14, 15, 16, 17, 18, 19, 20], which indicate their close relation with the fluid dynamics from membrane paradigm [21, 22, 23, 24, 25], as well as the fluid/gravity correspondence from holography [26, 27, 28, 29, 30].

It was noted in [2] that the nonlinear solution of vacuum Einstein gravity is of an algebraically special Petrov type [31, 32, 33], and the procedure was reversed via the near horizon expansion in [34] to derive the dual hydrodynamics. The Petrov type I condition

is imposed to reduce the Einstein equations to the incompressible Navier-Stokes equations in one lower dimension. The universal fixed-point behavior of the near-horizon scaling in general relativity is shown to be the same as that of hydrodynamic scaling in fluid dynamics [34]. This condition is expected to be equivalent with the regularity on the future horizon, and the framework has also been generalized to the highly accelerated surface which is spatially curved, and the case with the cosmology constant and Maxwell field in the bulk [35, 36, 37].

Note that in those works only the nontrivial leading order has been considered, we are here going to generalize the procedure to higher order to see whether the equivalence still holds or not. In the frame which is associated with a hypersurface where the dual fluid lived on, we find that the non-relativistic fluid dual to the non-linear solution of vacuum Einstein gravity from boost transformation does not satisfy the Petrov type I condition at the higher order, unless additional constraint is added such as the irrotational condition. We also inverse this procedure by imposing the Petrov type I condition on the fluid stress tensor, and then obtain the non-relativistic hydrodynamics with higher order corrections. But we see that the dual "Petrov type I fluid" can not match the dual fluid of vacuum Einstein gravity constructed in the non-relativistic limit. We study the procedure in two equivalent expansions: one is the non-relativistic hydrodynamic expansion associated with a finite cutoff surface, the other is the near horizon expansion associated with a highly accelerated surface.

This paper is organised as follows. In section 2, a simple review of the Petrov type I condition is given. In section 3, the higher order non-relativistic stress tensor dual to vacuum Einstein gravity is used to check the Petrov type I condition. Then the logic is turned around and the Petrov type I condition is imposed to reduce the gravity to the dual non-relativistic hydrodynamics. In section 4, an alternative presentation of this procedure in the near horizon expansion is discussed. The results and discussions are given in section 5.

2 Petrov type I condition

Firstly, we give a simple review of the Petrov type I condition with respect to the ingoing and outgoing pair of null vectors whose tangents to a timelike hypersurface generate time translations [34]. Introducing the $(p+2)$ Newman-Penrose-like vector fields,

$$\ell^2 = k^2 = 0, \quad (k,\ell) = 1, \quad (m_i, k) = (m_i, \ell) = 0, \quad (m_i, m_j) = \delta_{ij}, \tag{1}$$

the spacetime is Petrov type I [32, 33] if for some choice of frame,

$$C_{(\ell)i(\ell)j} = 0, \quad C_{(\ell)i(\ell)j} \equiv \ell^\mu m_i^\nu \ell^\alpha m_j^\beta C_{\mu\nu\alpha\beta}. \tag{2}$$

Consider a timelike $(p+1)$-dimensional hypersurface Σ_c with flat intrinsic metric

$$ds_{p+1}^2 = \gamma_{ab} dx^a dx^b = -(dx^0)^2 + \delta_{ij} dx^i dx^j, \quad i,j = 1, ..., p, \tag{3}$$

and extrinsic curvature K_{ab}. The hypersurface is embedded in $(p+2)$-dimensional vacuum Einstein spacetime that

$$G_{\mu\nu} = 0, \quad \mu, \nu = 0, ..., p+1. \tag{4}$$

Choosing the frame that

$$m_i = \partial_i, \ \ \sqrt{2}\ell = \partial_0 - n, \ \ \sqrt{2}k = -\partial_0 - n, \tag{5}$$

where n is the spacelike unit normal to the hypersurface, and ∂_i, ∂_0 are the tangent vectors to Σ_c [34], one has

$$2C_{(\ell)i(\ell)j} = (K - K_{00})K_{ij} + 2K_{0i}K_{0j} + 2\partial_0 K_{ij} - K_{ik}K^k_{\ j} - \partial_i K_{0j} - \partial_j K_{0i}, \tag{6}$$

where the following projections to Σ_c have been used

$$\begin{aligned}
\gamma^\alpha_a \gamma^\beta_b \gamma^\gamma_c \gamma^\delta_d C_{\alpha\beta\gamma\delta} &= K_{ad}K_{bc} - K_{ac}K_{bd}, \\
\gamma^\alpha_a \gamma^\beta_b \gamma^\gamma_c n^\delta C_{\alpha\beta\gamma\delta} &= \partial_a K_{bc} - \partial_b K_{ac}, \\
\gamma^\alpha_a n^\beta \gamma^\gamma_c n^\delta C_{\alpha\beta\gamma\delta} &= KK_{ac} - K_{ab}K^b_{\ c},
\end{aligned} \tag{7}$$

with $\gamma^\alpha_a = \delta^\alpha_a - n_a n^\alpha$. The Petrov type I condition (2) imposes $(p-1)(p+2)/2$ constraints on the $(p+1)(p+2)/2$ components of K_{ab}, or determines the trace-free part of K_{ij} in terms of K, K_{00} and K_{0i}. This leaves $(p+2)$ independent components, which are exactly the number of components of a fluid with a local energy density, pressure and velocity. The dual fluid is described by the Brown-York stress tensor on the hypersurface,

$$T_{ab} = 2(K\gamma_{ab} - K_{ab}). \tag{8}$$

The Hamiltonian constraint of vacuum Einstein equations

$$2G_{\mu\nu}n^\mu n^\nu|_{\Sigma_c} = (K^2 - K_{ab}K^{ab}) = 0 \Longrightarrow T^2 - p\,T_{ab}T^{ab} = 0, \tag{9}$$

can be viewed as the equation of state for the dual fluid relating the pressure and energy density. On the other hand, the $(p+1)$ momentum constraint equations

$$2G_{\mu b}n^\mu|_{\Sigma_c} = 2(\partial^a K_{ab} - \partial_b K) = 0 \Longrightarrow \partial^a T_{ab} = 0, \tag{10}$$

give us the equations of motion for the dual fluid.

3 On finite cutoff surface

In this section, with the non-relativistic stress tensor of fluid dual to vacuum Einstein gravity at finite cutoff surface given in [3], we will firstly check whether the Petrov type I condition is satisfied or not at higher order. Then we impose the Petrov type I condition to

reduce the gravity into the dual non-relativistic hydrodynamics. With the ingoing Rindler metric

$$\mathrm{d}s^2_{p+2} = -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i, \tag{11}$$

the induced metric at the finite cutoff surface $r = r_c$ is

$$\mathrm{d}s^2_{p+1} = \gamma_{ab}\mathrm{d}x_a\mathrm{d}x^b = -r_c\mathrm{d}\tau^2 + \mathrm{d}x_i\mathrm{d}x^i. \tag{12}$$

The Hamiltonian constraint becomes $\mathrm{H} = 0$, where

$$\mathrm{H} \equiv T^\tau{}_\tau T^\tau{}_\tau - 2r_c T^\tau{}_i T^\tau{}_j\delta^{ij} + T_{ij}T^{ij} - p^{-1}T^2. \tag{13}$$

Defining $\mathrm{P}_{ij} = 4C_{(\ell)i(\ell)j}$ and using equations (6) and (8), the Petrov type I condition turns out to be $\mathrm{P}_{ij} = 0$, where

$$\begin{aligned} 2\mathrm{P}_{ij} \equiv\, & T^\tau{}_\tau T_{ij} + 2r_c T^\tau{}_i T^\tau{}_j - 4r_c^{-1/2}\partial_\tau T_{ij} - T_{ik}T^k{}_j - 4r_c^{1/2}\partial_{(i}T^\tau{}_{j)} \\ & + p^{-2}\left[T\left(T - pT^\tau{}_\tau\right) + 4pr_c^{-1/2}\partial_\tau T\right]\delta_{ij}. \end{aligned} \tag{14}$$

3.1 Non-relativistic fluid and Petrov type I condition

Take the non-relativistic expansion in [2, 3]

$$v_i \sim \epsilon, \qquad P \sim \epsilon^2, \qquad \partial_i \sim \epsilon, \qquad \partial_\tau \sim \epsilon^2, \tag{15}$$

the Brown-York stress tensor up to order ϵ^4 is can be expressed as [3]

$$T^\tau{}_i = +\, r_c^{-3/2}v_i + r_c^{-5/2}\left[v_i(v^2+P) - 2r_c\sigma_{ij}v^j\right] + O(\epsilon^5), \tag{16}$$

$$T^\tau{}_\tau = -\, r_c^{-3/2}v^2 - r_c^{-5/2}\left[v^2(v^2+P) - 2r_c\sigma_{ij}v^iv^j - 2r_c^2\sigma_{ij}\sigma^{ij}\right] + O(\epsilon^6), \tag{17}$$

$$\begin{aligned} T_{ij} =\, & +\, r_c^{-1/2}\,\delta_{ij} + r_c^{-3/2}\left[P\delta_{ij} + v_iv_j - 2r_c\sigma_{ij}\right] \\ & + r_c^{-5/2}\left[v_iv_j(v^2+P) - r_c\sigma_{ij}v^2 + 2r_cv_{(i}\partial_{j)}P - r_cv_{(i}\partial_{j)}v^2 - 2r_c^2v_{(i}\partial^2v_{j)}\right. \\ & \left. -2r_c^2\sigma_{ik}\sigma^k{}_j - 4r_c^2\sigma_{k(i}\omega^k{}_{j)} - 4r_c^2\omega_{ik}\omega^k{}_j - 4r_c^2\partial_i\partial_jP + 3r_c^3\partial^2\sigma_{ij}\right] + O(\epsilon^6), \end{aligned} \tag{18}$$

$$T\ = T^\tau{}_\tau + T^i{}_i = p\,r_c^{-1/2} + p\,r_c^{-3/2}P + O(\epsilon^6), \tag{19}$$

where the fluid shear σ_{ij} and vorticity ω_{ij} are given by [1]

$$\sigma_{ij} \equiv \partial_{(i}v_{j)} = \left(\partial_iv_j + \partial_jv_i\right)/2, \qquad \omega_{ij} \equiv \partial_{[i}v_{j]} = \left(\partial_iv_j - \partial_jv_i\right)/2. \tag{20}$$

Comparing this stress tensor with the non-relativistic fluid stress tensor given in Appendix B.1, one can read off some transport coefficients as

$$\eta = 1, \quad c_1 = -2, \quad c_2 = c_3 = c_4 = -4. \tag{21}$$

[1]Here the notations are different from [3] with a factor 2

The equations of motion of the dual fluid $\partial^a T_{ab} = 0$ turn out to be the incompressible Navier-Stokes equations with higher order corrections given in (79), and the stress tensor satisfies the Hamiltonian constraint $\mathrm{H} = 0$ consistently. Inserting the stress tensor (16)-(19) into P_{ij} and expanding in powers of parameter ϵ, one has

$$\mathrm{P}_{ij} = \mathrm{P}_{ij}^{(0)} + \mathrm{P}_{ij}^{(2)} + \mathrm{P}_{ij}^{(4)} + O(\epsilon^6). \tag{22}$$

Taking into account the equations of motion (79), one can see that $\mathrm{P}_{ij}^{(0)}$ and $\mathrm{P}_{ij}^{(2)}$ vanish identically, but

$$\mathrm{P}_{ij}^{(4)} = r_c^{-3}\left[-6r_c v^k v_{(i}\omega_{j)k} - 2r_c^2 v_{(i}\partial^2 v_{j)} + 4r_c^2 v^k \partial_{(i}\omega_{j)k} + r_c^3 \partial^2 \sigma_{ij}\right]. \tag{23}$$

This result can also be obtained through substituting the nonlinear solution of vacuum Einstein gravity given in Appendix A.1 into the Weyl tensor (2) directly. And it is independent of the gauge transformation that $v_i \to v_i + \delta v_i$ or $T_{ij} \to T_{ij} + \delta P \delta_{ij}$, where $\delta v_i \sim \epsilon^3$, $\delta P \sim \epsilon^4$. Thus the perturbed stress tensor (16)-(19) on the finite cutoff surface does not satisfy the Petrov type I condition at order ϵ^4, if we choose this frame (5) associated with the finite cutoff hypersurface. Or in other words, the non-linear solution of vacuum Einstein gravity constructed by boost transformation, up to order ϵ^4, does not satisfy the Petrov type I condition.

But we can additionally require the constraint $\mathrm{P}_{ij}^{(4)} = 0$ holds. For example, if we take the irroational condition with $\omega_{ij} \sim O(\epsilon^4)$, then in view of $\theta \equiv \partial_i v^i \sim O(\epsilon^4)$, one has

$$\partial^2 v_j = \partial_j \theta - 2\partial^k \omega_{jk} \sim O(\epsilon^5), \tag{24}$$

$$\partial^2 \sigma_{ij} = \partial_{(i}\partial_{j)}\theta - 2\partial^k \partial_{(i}\omega_{j)k} \sim O(\epsilon^6). \tag{25}$$

Thus $\mathrm{P}_{ij}^{(4)}$ vanishes at this order and T_{ij} is reduced to

$$\begin{aligned} T_{ij}^{(\sigma)} = {} & r_c^{-1/2}\,\delta_{ij} + r_c^{-3/2}\left[P\delta_{ij} + v_i v_j - 2r_c\sigma_{ij}\right] + r_c^{-5/2}\big[v_i v_j(v^2 + P) \\ & - r_c\sigma_{ij}v^2 + 2r_c v_{(i}\partial_{j)}P - r_c v_{(i}\partial_{j)}v^2 - 2r_c^2\sigma_{ik}\sigma^k{}_j - 4r_c^2\partial_i\partial_j P\big]. \end{aligned} \tag{26}$$

In this case, comparing (26) with the non-relativistic fluid stress tensor in Appendix B.1, we can read off

$$\eta = 1, \quad c_1 = -2, \quad c_4 = -4. \tag{27}$$

The incompressible Navier-Stokes equations with higher order corrections (79) is reduced into

$$\partial_i v^i = \theta^{(\sigma)}, \qquad \partial_\tau v_i + v^j \partial_j v_i + \partial_i P = r_c \partial^2 v_i + f_i^{(\sigma)}, \tag{28}$$

where the higher order corrections become

$$\theta^{(\sigma)} = + 2\sigma_{ij}\sigma^{ij} + r_c^{-1} v^i \partial_i P + O(\epsilon^6), \tag{29}$$

$$\begin{aligned} f_i^{(\sigma)} = {} & - 3r_c\partial_i(\sigma_{kl}\sigma^{kl}) + 4r_c\sigma^{kl}\partial_k\sigma_{li} - 2v^k\partial_k\partial_i P - 2(\partial^k v_i)\partial_k P \\ & - (\partial_k\sigma_{il})v^k v^l + r_c^{-1}(P + v^2)\partial_i P - r_c^{-1} v_i \partial_\tau P + O(\epsilon^7). \end{aligned} \tag{30}$$

Here according to (24), the term $r_c\partial^2 v_i \sim O(\epsilon^5)$, therefore we move this term to the right hand side of the Navier-Stokes equations in (28).

3.2 From Petrov type I condition to dual fluid

At the finite cutoff surface, if we impose the Petrov type I condition $\mathrm{P}_{ij} = 0$ firstly, and consider the non-relativistic hydrodynamic scaling laws in (15), then the Brown-York stress tensor can be expanded in powers of the non-relativistic hydrodynamic expansion parameter ϵ as

$$\begin{aligned} T^{\tau}{}_{i} &= T^{\tau}{}_{i}^{(1)} + T^{\tau}{}_{i}^{(3)} + O(\epsilon^5), \\ T^{\tau}{}_{\tau} &= T^{\tau}{}_{\tau}^{(0)} + T^{\tau}{}_{\tau}^{(2)} + T^{\tau}{}_{\tau}^{(4)} + O(\epsilon^6), \\ T_{ij} &= T_{ij}^{(0)} + T_{ij}^{(2)} + T_{ij}^{(4)} + O(\epsilon^6), \\ T &= T^{(0)} + T^{(2)} + T^{(4)} + O(\epsilon^6). \end{aligned} \tag{31}$$

Here superscript in round brackets stands for the expansion order, such as $T^{\tau}{}_{i}^{(1)} \sim \epsilon$, $T^{\tau}{}_{i}^{(3)} \sim \epsilon^3$, and so on. The Brown-York stress tensor at the cutoff surface $r = r_c$ of the metric (11) gives

$$T^{\tau}{}_{\tau}^{(0)} = 0, \qquad T_{ij}^{(0)} = r_c^{-1/2}\,\delta_{ij}, \quad T^{(0)} = r_c^{-1/2}\,p\,. \tag{32}$$

We now put the expansions (31) into the Hamiltonian constraint equation (13) and the Petrov equations (14), which both can be expanded in powers of the parameter ϵ. The first non-trivial order appears at order ϵ^2, where the Hamiltonian constraint $\mathrm{H}^{(2)} = 0$ and Petrov type I condition $\mathrm{P}_{ij}^{(2)} = 0$ lead to

$$T^{\tau}{}_{\tau}^{(2)} = -\,T^{\tau}{}_{i}^{(1)} T^{\tau}{}_{j}^{(1)} \delta^{ij}, \tag{33}$$

$$T_{ij}^{(2)} = p^{-1} T^{(2)} \delta_{ij} + r_c^{3/2}\, T^{\tau}{}_{i}^{(1)} T^{\tau}{}_{j}^{(1)} - 2\, r_c\, \partial_{(i} T^{\tau}{}_{j)}^{(1)}, \tag{34}$$

respectively. Following [34], if we assume that

$$T^{\tau}{}_{i}^{(1)} = r_c^{-3/2} v_i, \quad T^{(2)} = r_c^{-3/2} p\, P, \tag{35}$$

we can recover the stress tensor (16)-(19) up to order ϵ^2. The next non-trivial Hamiltonian constraint $\mathrm{H}^{(4)} = 0$ and Petrov type I condition $\mathrm{P}_{ij}^{(4)} = 0$ give

$$T^{\tau}{}_{\tau}^{(4)} = -\, r_c^{3/2} T^{\tau}{}_{i}^{(1)} T^{\tau}{}_{j}^{(3)} \delta^{ij} + \frac{1}{2}\left[r_c^{1/2} T_{ij}^{(2)} T^{ij}_{(2)} + r_c^{1/2} (T^{\tau}{}_{\tau}^{(2)})^2 - p^{-1} r_c^{1/2} (T^{(2)})^2 \right], \tag{36}$$

$$\begin{aligned} T_{ij}^{(4)} =&\ 2\, r_c^{3/2} T^{\tau}{}_{(i}^{(1)} T^{\tau}{}_{j)}^{(3)} - 2 r_c\, \partial_{(i} T^{\tau}{}_{j)}^{(3)} + \frac{1}{2} r_c^{1/2} T^{\tau}{}_{\tau}^{(2)} T_{ij}^{(2)} - \frac{1}{2} T_{ik}^{(2)} T_{jl}^{(2)} \delta^{kl} - \partial_\tau T_{ij}^{(2)} \\ &+ \frac{1}{2}\, p^{-1} \left[p^{-1} r_c^{1/2} (T^{(2)})^2 - r_c^{1/2} T^{(2)} T^{\tau}{}_{\tau}^{(2)} + 4 \partial_\tau T^{(2)} + 2\, T^{(4)} \right] \delta_{ij}, \end{aligned} \tag{37}$$

respectively. To give assumptions at higher orders, we choose the Landau frame which gives

$$0 = h_a^b T_{bc} u^c, \quad h_a^b = \delta_a^b + u_a u^b, \tag{38}$$

where $u^a = \gamma_v (1, v^i)$ and $\gamma_{ab} u^a u^b = -1$ [3]. At order ϵ^3, its spatial components lead to

$$0 = -r_c T^{\tau}{}_{i}^{(3)} + T_{ij}^{(2)} v^j + \mathrm{e}^{(2)} v_i, \tag{39}$$

where the energy density $\mathrm{e} \equiv T_{ab}u^a u^b$. With the recovered stress tensor up to ϵ^2, one can show $\mathrm{e}^{(2)} = 0$. Putting (34) and (35) into the above equation, we obtain

$$T^{\tau}{}_{i}^{(3)} = r_c^{-5/2}\left[v_i(v^2+P) - 2r_c\sigma_{ij}v^j\right]. \tag{40}$$

Then $T^\tau{}_\tau$ in (17) can be recovered up to order ϵ^4 with the Hamiltonian constraint which leads to (33) and (36). On the other hand, putting (34) (35) and (40) into (37), one finds that at order ϵ^4, there is only one term $T^{(4)}\delta_{ij}$ proportional to δ_{ij}. Thus, we can choose the isotropic gauge with $T^{(4)} = 0$ as in [3], and finally $T_{ij}^{(4)}$ is given by

$$\begin{aligned}T_{ij}^{(4)} =& r_c^{-5/2}\big[v_iv_j(v^2+P) - r_c\sigma_{ij}v^2 + 2r_cv_{(i}\partial_{j)}P - r_cv_{(i}\partial_{j)}v^2 + 6r_cv_kv_{(i}\omega^k{}_{j)} - 4r_c^2v_{(i}\partial^2v_{j)}\\ &-2r_c^2\sigma_{ik}\sigma^k{}_j - 4r_c^2\sigma_{k(i}\omega^k{}_{j)} - 4r_c^2\omega_{ik}\omega^k{}_j - 4r_c^2\partial_i\partial_jP - 4r_c^2v_k\partial_{(i}\omega^k{}_{j)} + 4r_c^3\partial^2\sigma_{ij}\big].\end{aligned} \tag{41}$$

Compare (41) with the terms in (18) at order ϵ^4, we obtain the additional terms

$$r_c^{-5/2}\left[6r_cv_kv_{(i}\omega^k{}_{j)} - 2r_c^2v_{(i}\partial^2v_{j)} - 4r_c^2v_k\partial_{(i}\omega^k{}_{j)} + r_c^3\partial^2\sigma_{ij}\right]. \tag{42}$$

Thus, the incompressible Navier-Stokes equations with higher order corrections from the equations of motion of the fluid $\partial^aT_{ab} = 0$ become

$$\partial_iv^i = \theta, \qquad \partial_\tau v_i + v^j\partial_jv_i - r_c\partial^2v_i + \partial_iP = f_i + f_i^{(\omega)}, \tag{43}$$

where θ and f_i are given in (80) and (81), respectively, and

$$\begin{aligned}f_i^{(\omega)} =& -\frac{r_c^2}{2}\partial^4v_i + 4r_cv^k\partial^2\omega_{ki} + 2r_c\partial_l\omega_{ki}\partial^lv^k + 2r_c\partial_kv_i\partial_l\omega^{lk} + r_c\partial_i(\omega_{kl}\omega^{lk})\\ &- 3v_i(\omega_{kl}\omega^{lk}) - 3v_iv_k\partial_l\omega^{kl} - 3v_k\omega_{li}\partial^kv^l - 3v_k(\partial_lv_i)\omega^{kl} - 3(\partial_l\omega_{ki})v^kv^l + O(\epsilon^7).\end{aligned} \tag{44}$$

Comparing (41) with the non-relativistic fluid dual to vacuum Einstein gravity constructed in Appendix B.1, one can extract the second order transport coefficients as

$$c_1 = -2, \quad c_2 = c_3 = c_4 = -4, \tag{45}$$

which implies that the correction terms in (42) do not contribute to the terms associated with second order transport coefficients. Thus, such kind of higher order fluid reduced from the Petrov type I condition, which we name as "Petrov type I fluid", does not satisfy the non-relativistic fluid that constructed in Appendix B.1. However, if additionally requiring that the terms in (42) vanish at this order, we can again recover the previous stress tensor (16)-(19), up to order ϵ^4. In particular, taking the irrotational condition that $\omega_{ij} \sim O(\epsilon^4)$, we can recover equations (26)-(30).

4 On highly accelerated surface

An alternative presentation of the procedure discussed in the previous section can also be realized with the near horizon expansion. Introducing the expansion parameter $\lambda = r_c^{1/2}$

via the transformation $\tau \to \lambda^{-2}\hat{\tau},\ r \to \lambda^2\hat{r},\ x \to \hat{x}$, the ingoing Rindler metric (11) becomes

$$d\hat{s}^2_{p+2} = -\frac{\hat{r}}{\lambda^2}d\hat{\tau}^2 + 2d\hat{\tau}d\hat{r} + d\hat{x}_i d\hat{x}^i, \tag{46}$$

which gives the first three terms in (84). The induced metric (12) changes into

$$d\hat{s}^2_{p+1} = \hat{\gamma}_{ab}d\hat{x}^a d\hat{x}^b = -\frac{1}{\lambda^2}d\hat{\tau}^2 + d\hat{x}_i d\hat{x}^i. \tag{47}$$

In the hatted coordinates, the Hamiltonian constraint becomes $\hat{\mathrm{H}} = 0$, where

$$\hat{\mathrm{H}} \equiv \hat{T}^{\hat{\tau}}{}_{\hat{\tau}}\hat{T}^{\hat{\tau}}{}_{\hat{\tau}} - 2\lambda^{-2}\hat{T}^{\hat{\tau}}{}_i\hat{T}^{\hat{\tau}}{}_j\delta^{ij} + \hat{T}_{ij}\hat{T}^{ij} - p^{-1}\hat{T}^2. \tag{48}$$

The Petrov type I condition turns out to be $\hat{\mathrm{P}}_{ij} = 0$, where

$$\begin{aligned}2\hat{\mathrm{P}}_{ij} \equiv\ & \hat{T}^{\hat{\tau}}{}_{\hat{\tau}}\hat{T}_{ij} + 2\lambda^{-2}\hat{T}^{\hat{\tau}}{}_i\hat{T}^{\hat{\tau}}{}_j - 4\lambda\,\hat{\partial}_{\hat{\tau}}\hat{T}_{ij} - \hat{T}_{ik}\hat{T}^k{}_j - 4\lambda^{-1}\hat{\partial}_{(i}T^{\hat{\tau}}{}_{j)} \\ &+ p^{-2}\left[\hat{T}(\hat{T} - p\hat{T}^{\hat{\tau}}{}_{\hat{\tau}}) + 4p\lambda\,\partial_{\hat{\tau}}\hat{T}\right]\delta_{ij}.\end{aligned} \tag{49}$$

4.1 Near horizon fluid and Petrov type I condition

In the near horizon expansion, with the transformations (82),(83) and (99), the stress tensor (16)-(19) becomes

$$\hat{T}^{\hat{\tau}}{}_i = +\,\lambda v_i + \lambda^3\left[\hat{v}_i(\hat{v}^2 + \hat{P}) - 2\hat{\sigma}_{ij}\hat{v}^j\right] + O(\lambda^5), \tag{50}$$

$$\hat{T}^{\hat{\tau}}{}_{\hat{\tau}} = -\,\lambda v^2 - \lambda^3\left[\hat{v}^2(\hat{v}^2 + \hat{P}) - 2\hat{\sigma}_{ij}\hat{v}^i\hat{v}^j - 2\hat{\sigma}_{ij}\hat{\sigma}^{ij}\right] + O(\lambda^5), \tag{51}$$

$$\begin{aligned}\hat{T}_{ij} =\ & +\,\lambda^{-1}\,\delta_{ij} + \lambda\left[\hat{P}\delta_{ij} + \hat{v}_i\hat{v}_j - 2\hat{\sigma}_{ij}\right] \\ &+ \lambda^3\Big[\hat{v}_i\hat{v}_j(\hat{v}^2 + \hat{P}) - \hat{\sigma}_{ij}\hat{v}^2 + 2\hat{v}_{(i}\hat{\partial}_{j)}\hat{P} - \hat{v}_{(i}\hat{\partial}_{j)}\hat{v}^2 - 2\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} \\ &-2\hat{\sigma}_{ik}\hat{\sigma}^k{}_j - 4\hat{\sigma}_{k(i}\hat{\omega}^k{}_{j)} - 4\hat{\omega}_{ik}\hat{\omega}^k{}_j - 4\hat{\partial}_i\hat{\partial}_j\hat{P} + 3\hat{\partial}^2\hat{\sigma}_{ij}\Big] + O(\lambda^5),\end{aligned} \tag{52}$$

$$\hat{T}\ = \hat{T}^{\hat{\tau}}{}_{\hat{\tau}} + \hat{T}^i{}_i = \lambda^{-1}p + \lambda\,p\,P + O(\lambda^5), \tag{53}$$

where the fluid shear $\hat{\sigma}_{ij} \equiv \hat{\partial}_{(i}\hat{v}_{j)}$ and vorticity $\hat{\omega}_{ij} \equiv \hat{\partial}_{[i}\hat{v}_{j]}$. Comparing the stress tensor with the one of dual fluid given in Appendix B.2, one has

$$\hat{\eta} = 1, \quad \hat{c}_1 = -2, \quad \hat{c}_2 = \hat{c}_3 = \hat{c}_4 = -4. \tag{54}$$

The equations of motion $\hat{\partial}^a\hat{T}_{ab} = 0$ turn out to be (86), and the stress tensor satisfies the Hamiltonian constraint $\hat{\mathrm{H}} = 0$ consistently. Inserting equations (50)-(53) into $\hat{\mathrm{P}}_{ij}$ with expansion in powers of λ, we have

$$\hat{\mathrm{P}}_{ij} = \lambda^{-2}\hat{\mathrm{P}}^{(-2)}_{ij} + \lambda^0\hat{\mathrm{P}}^{(0)}_{ij} + \lambda^2\hat{\mathrm{P}}^{(2)}_{ij} + O(\lambda^4). \tag{55}$$

We see that $\hat{\mathrm{P}}^{(-2)}_{ij}$ and $\hat{\mathrm{P}}^{(0)}_{ij}$ vanish identically, but

$$\hat{\mathrm{P}}^{(2)}_{ij} = -6\hat{v}^k\hat{v}_{(i}\hat{\omega}_{j)k} - 2\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} + 4\hat{v}^k\hat{\partial}_{(i}\hat{\omega}_{j)k} + \hat{\partial}^2\hat{\sigma}_{ij}. \tag{56}$$

This is independent of the gauge transformation with $\hat{v}_i \to \hat{v}_i + \lambda^2\delta\hat{v}_i$ or $\hat{T}_{ij} \to \hat{T}_{ij} + \lambda^3\delta\hat{P}\delta_{ij}$. Thus the perturbed stress tensor (50)-(53) does not satisfy the Petrov type I condition at order λ^2, if we choose this frame (5).

Again, we can also additionally require $\hat{\mathrm{P}}^{(2)}_{ij} = 0$. For example, if we add the irroational condition that $\hat{\omega}_{ij} \sim O(\lambda^2)$, then $\hat{\mathrm{P}}^{(2)}_{ij}$ vanishes at this order and $\hat{T}_{ij}$ is reduced to

$$\begin{aligned}\hat{T}^{(\hat{\sigma})}_{ij} = \lambda^{-1}\,\delta_{ij} + \lambda\left[\hat{P}\delta_{ij} + \hat{v}_i\hat{v}_j - 2\hat{\sigma}_{ij}\right] + \lambda^3\Big[\hat{v}_i\hat{v}_j(\hat{v}^2 + \hat{P})\\ -\hat{\sigma}_{ij}\hat{v}^2 + 2\hat{v}_{(i}\hat{\partial}_{j)}\hat{P} - \hat{v}_{(i}\hat{\partial}_{j)}\hat{v}^2 - 2\hat{\sigma}_{ik}\hat{\sigma}^k_j - 4\hat{\partial}_i\hat{\partial}_j P\Big].\end{aligned} \tag{57}$$

Comparing this with the stress tensor of dual fluid given in Appendix B.2, we have

$$\hat{\eta} = 1, \quad \hat{c}_1 = -2, \quad \hat{c}_4 = -4. \tag{58}$$

In this case, the incompressible Navier-Stokes equations with higher order corrections (86) is reduced to

$$\hat{\partial}_i\hat{v}^i = \hat{\theta}^{(\hat{\sigma})}, \quad \partial_{\hat{\tau}}\hat{v}_i + \hat{v}^j\hat{\partial}_j\hat{v}_i + \hat{\partial}_i\hat{P} = \hat{\partial}^2\hat{v}_i + \hat{f}^{(\hat{\sigma})}_i, \tag{59}$$

where the higher order corrections are given by

$$\hat{\theta}^{(\hat{\sigma})} = \lambda^2\left[+2\hat{\sigma}_{ij}\hat{\sigma}^{ij} + \hat{v}^i\hat{\partial}_i\hat{P}\right] + O(\lambda^4), \tag{60}$$

$$\begin{aligned}\hat{f}^{(\hat{\sigma})}_i = \lambda^2\Big[-3\hat{\partial}_i(\hat{\sigma}_{kl}\hat{\sigma}^{kl}) + 4\hat{\sigma}^{kl}\hat{\partial}_k\hat{\sigma}_{li} - 2\hat{v}^k\hat{\partial}_k\hat{\partial}_i\hat{P} - 2(\hat{\partial}^k\hat{v}_i)\hat{\partial}_k\hat{P}\\ - (\hat{\partial}_k\hat{\sigma}_{il})\hat{v}^k\hat{v}^l + (\hat{P} + \hat{v}^2)\hat{\partial}_i\hat{P} - \hat{v}_i\hat{\partial}_{\hat{\tau}}\hat{P}\Big] + O(\lambda^4).\end{aligned} \tag{61}$$

Since the term $\hat{\partial}^2\hat{v}_i \sim O(\lambda^2)$, it is therefore put on the right hand side of the equation (59).

4.2 From Petrov type I condition to dual fluid

In this subsection we will inverse the procedure and expand the Brown-York stress tensor in powers of the parameter λ with the background metric (47),

$$\begin{aligned}\hat{T}^{\hat{\tau}}_{\ i} &= \lambda\,\hat{T}^{\hat{\tau}(1)}_{\ i} + \lambda^3\,\hat{T}^{\hat{\tau}(3)}_{\ i} + O(\lambda^5),\\ \hat{T}^{\hat{\tau}}_{\ \hat{\tau}} &= \lambda\,\hat{T}^{\hat{\tau}(1)}_{\ \hat{\tau}} + \lambda^3\,\hat{T}^{\hat{\tau}(3)}_{\ \hat{\tau}} + O(\lambda^5),\\ \hat{T}_{ij} &= \lambda^{-1}\delta_{ij} + \lambda\,\hat{T}^{(1)}_{ij} + \lambda^3\,\hat{T}^{(3)}_{ij} + O(\lambda^5),\\ \hat{T} &= \lambda^{-1}p + \lambda\,\hat{T}^{(1)} + \lambda^3\,\hat{T}^{(3)} + O(\lambda^5).\end{aligned} \tag{62}$$

Note that here only the odd order terms are selected. The even order terms can also be added, because it can be showed that they give no further information of the higher order

fluid, and thus are set to be vanished to satisfy the constraint equations as well as Petrov type I condition. We now put the expansions (62) into the Hamiltonian equation (48) and the Petrov equations (49), which both can be expanded in powers of the parameter λ. The first non-trivial order appears ar λ^0, where the Hamiltonian constraint $\hat{\mathrm{H}}^{(0)}=0$ and Petrov type I condition $\hat{\mathrm{P}}^{(0)}_{ij}=0$ lead to

$$\hat{T}^{\hat{\tau}(1)}_{\hat{\tau}} = -\,\hat{T}^{\tau(1)}_{\ i}\hat{T}^{\tau(1)}_{\ j}\delta^{ij}, \tag{63}$$

$$\hat{T}^{(1)}_{ij} = p^{-1}\hat{T}^{(1)}\delta_{ij} + \hat{T}^{\hat{\tau}(1)}_{\ i}\hat{T}^{\hat{\tau}(1)}_{\ j} - 2\,\hat{\partial}_{(i}\hat{T}^{\hat{\tau}(1)}_{\ j)}, \tag{64}$$

respectively. Again, following [34], if assuming that

$$\hat{T}^{\tau(1)}_{\ i} = \hat{v}_i, \qquad \hat{T}^{(1)} = p\,\hat{P}, \tag{65}$$

we can recover the stress tensor (50)-(53) up to order λ. The next non-trivial Hamiltonian constraint $\hat{\mathrm{H}}^{(2)}=0$ and Petrov type I condition $\hat{\mathrm{P}}^{(2)}_{ij}=0$ give

$$\hat{T}^{\hat{\tau}(3)}_{\hat{\tau}} = -\,\hat{T}^{\tau(1)}_{\ i}\hat{T}^{\tau(3)}_{\ j}\delta^{ij} + \frac{1}{2}\left[\hat{T}^{(1)}_{ij}\hat{T}^{ij}_{(1)} + (\hat{T}^{\hat{\tau}(1)}_{\hat{\tau}})^2 - p^{-1}(\hat{T}^{(1)})^2\right], \tag{66}$$

$$\begin{aligned}\hat{T}^{(3)}_{ij} = {} & 2\,\hat{T}^{\tau(1)}_{\ (i}\hat{T}^{\tau(3)}_{\ j)} - 2\,\hat{\partial}_{(i}\hat{T}^{\hat{\tau}(3)}_{\ j)} + \frac{1}{2}\hat{T}^{\hat{\tau}(1)}_{\hat{\tau}}\hat{T}^{(1)}_{ij} - \frac{1}{2}\hat{T}^{(1)}_{i\,k}\hat{T}^{(1)}_{j\,l}\delta^{kl} - 2\hat{\partial}_{\hat{\tau}}\hat{T}^{(1)}_{ij} \\ & + \frac{1}{2}\,p^{-1}\left[p^{-1}(\hat{T}^{(1)})^2 - \hat{T}^{(1)}\hat{T}^{\hat{\tau}(1)}_{\tau} + 4\hat{\partial}_{\hat{\tau}}\hat{T}^{(1)} + 2\,\hat{T}^{(3)}\right]\delta_{ij},\end{aligned} \tag{67}$$

respectively. To give assumptions at higher order, we choose the Landau frame which gives

$$0 = \hat{h}^{b}_{a}\hat{T}_{bc}\hat{u}^c, \qquad \hat{h}^b_a = \delta^b_a + \hat{u}_a\hat{u}^b, \tag{68}$$

where $\hat{u}^a = \hat{\gamma}_v(1,\hat{v}^i)$ and $\hat{\gamma}_{ab}\hat{u}^a\hat{u}^b = -1$. At order λ, the spatial components give us with

$$0 = -\hat{T}^{\hat{\tau}(3)}_{\ i} + \hat{T}^{(1)}_{ij}\hat{v}^j + \hat{\mathrm{e}}^{(1)}v_i, \tag{69}$$

where $\hat{\mathrm{e}} \equiv \hat{T}_{ab}\hat{u}^a\hat{u}^b$. From the recovered stress tensor up to order λ we have $\mathrm{e}^{(1)}=0$. Putting (64) and (65) into the above equation we get

$$\hat{T}^{\tau(3)}_{\ i} = \hat{v}_i(\hat{v}^2 + \hat{P}) - 2\hat{\sigma}_{ij}\hat{v}^j. \tag{70}$$

Then $\hat{T}^{\hat{\tau}}_{\ \tau}$ in (51) can be recovered up to order λ^3 via the Hamiltonian constraint which leads to (63) and (66). On the other hand, putting (64)(65) and (70) into (67), one finds that and at order λ^3, there is only one term $\hat{T}^{(3)}\delta_{ij}$ proportional to δ_{ij}. Thus, we can choose the isotropic gauge so that $\hat{T}^{(3)}=0$ and $\hat{T}^{(3)}_{ij}$ can be expressed as

$$\begin{aligned}\hat{T}^{(3)}_{ij} = {} & \hat{v}_i\hat{v}_j(\hat{v}^2+\hat{P}) - \hat{\sigma}_{ij}\hat{v}^2 + 2\hat{v}_{(i}\hat{\partial}_{j)}\hat{P} - \hat{v}_{(i}\hat{\partial}_{j)}\hat{v}^2 + 6\hat{v}_k\hat{v}_{(i}\hat{\omega}^k_{\ j)} - 4\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} \\ & - 2\hat{\sigma}_{ik}\hat{\sigma}^k_{\ j} - 4\hat{\sigma}_{k(i}\hat{\omega}^k_{\ j)} - 4\hat{\omega}_{ik}\hat{\omega}^k_{\ j} - 4\hat{\partial}_i\hat{\partial}_j\hat{P} - 4\hat{v}_k\hat{\partial}_{(i}\hat{\omega}^k_{\ j)} + 4\hat{\partial}^2\hat{\sigma}_{ij}.\end{aligned} \tag{71}$$

Comparing (71) with the terms in (52) at order λ^3, one can find that the additional terms are

$$6\hat{v}_k\hat{v}_{(i}\hat{\omega}^k{}_{j)} - 2\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} - 4\hat{v}_k\hat{\partial}_{(i}\hat{\omega}^k{}_{j)} + \hat{\partial}^2\hat{\sigma}_{ij}. \tag{72}$$

Thus, the incompressible Navier-Stokes equations with higher order corrections from the equations of motion of the fluid $\hat{\partial}^a\hat{T}_{ab} = 0$ become

$$\hat{\partial}_i\hat{v}^i = \hat{\theta}, \quad \partial_{\hat{\tau}}\hat{v}_i + \hat{v}^j\hat{\partial}_j\hat{v}_i - \hat{\partial}^2\hat{v}_i + \hat{\partial}_i\hat{P} = \hat{f}_i + \hat{f}_i^{(\hat{\omega})}, \tag{73}$$

where $\hat{\theta}$ and $\hat{f}_i$ are given in (87) and (88), respectively, and

$$\begin{aligned}\hat{f}_i^{(\hat{\omega})} =& \lambda^2 \Big[-\frac{1}{2}\hat{\partial}^4\hat{v}_i + 4\hat{v}^k\hat{\partial}^2\hat{\omega}_{ki} + 2\hat{\partial}_l\hat{\omega}_{ki}\hat{\partial}^l\hat{v}^k + 2\hat{\partial}_k\hat{v}_i\hat{\partial}_l\hat{\omega}^{lk} + \hat{\partial}_i(\hat{\omega}_{kl}\hat{\omega}^{lk}) - 3\hat{v}_i(\hat{\omega}_{kl}\hat{\omega}^{lk}) \\ & - 3\hat{v}_i\hat{v}_k\hat{\partial}_l\hat{\omega}^{kl} - 3\hat{v}_k\hat{\omega}_{li}\hat{\partial}^k\hat{v}^l - 3\hat{v}_k(\hat{\partial}_l\hat{v}_i)\hat{\omega}^{kl} - 3(\hat{\partial}_l\hat{\omega}_{ki})\hat{v}^k\hat{v}^l \Big] + O(\lambda^4).\end{aligned} \tag{74}$$

Comparing (71) with the stress tensor of fluid given in Appendix B.2, one can obtain the second order transport coefficients as

$$\hat{c}_1 = -2, \quad \hat{c}_2 = \hat{c}_3 = \hat{c}_4 = -4. \tag{75}$$

Thus, we have shown that the additional corrections do not make contribution to the second order transport coefficients. Such kind of higher order Petrov type I non-relativistic fluid does not match the fluid constructed in Appendix B.2. However, if we additionally require that the terms in (72) vanishes at this order, the stress tensor (50)-(53) can be recovered. In particular, taking the irrotational condition with $\hat{\omega}_{ij} \sim O(\lambda^2)$, we can still recover equations (57)-(61).

5 Conclusion

In Einstein gravity, the Petrov type I condition relate the gravity theory to a dual fluid without gravity in one less dimension. It reduces the the extrinsic curvature of a time-like hypersurface to $p+2$ components, which can be interpreted as the energy density, pressure P and velocity field v^i of a dual fluid living on the hypersurface, constrained by equation of state and $p+1$ evolution equations (incompressible Navier-Stokes equations) that come from the Einstein constraint equations [34]. To the non-linear order there are two equivalent presentations, that for the dual fluid living on a finite cutoff surface via non-relativistic hydrodynamic expansion, and on a highly accelerated surface via the near horizon expansion. Imposing the Petrov condition is mathematically much simpler than imposing regularity on the future horizon.

Via appropriate gauge choice, we generalized this procedure to the next order and obtained the incompressible Navier-Stokes equations with higher order corrections and associated second order transport coefficients. More higher order hydrodynamics can also be obtained order by order with appropriate expansion parameters. We can recover the non-relativistic fluid stress tensor dual to vacuum Einstein gravity from boost transformation

up to order ϵ^4, only if by imposing additional constraint such as the irrotational condition. In other words, the non-linear solution of vacuum Einstein equations constructed by boost transformation does not satisfy the Petrov type I condition up to order ϵ^4, although it holds at the order ϵ^2.

As the dual fluid constructed in Appendix B is reduced from the relativistic hydrodynamics, while the Petrov type I condition singles out a preferred time coordinate and thus breaks Lorentz invariance of the hypersurface [34]. Thus, it might be not surprised that the "Petrov type I fluid" does not match the boosted fluid at higher order. In this sense it would be interesting to construct the non-relativistic hydrodynamics of this special higher order fluid directly, with the corresponding non-linear gravitational solutions. Choosing a different frame instead of (5) or changing the boundary condition of the hypersurface to see their effects at higher orders, and generalizing to other bulk geometries would be interesting for further works.

Note added: During the preparation of this work, we were informed that the leading order calculation in section 3.2 might overlap with the work in preparation by authors in [37]. After finishing this work, we were informed that the authors in [3] also obtained the conclusion that the Petrov type I condition does not hold at higher orders for their non-linear solution of vacuum Einstein gravity (unpublished, May 2011).

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A Nonlinear metric solution

In this Appendix, we briefly give the nonlinear solution of vacuum Einstein equations $G_{\mu\nu} = 0$, which is obtained via the non-relativistic hydrodynamic expansion and near horizon expansion, respectively [2, 3].

A.1 Non-relativistic hydrodynamic expansion

Associated with the non-relativistic hydrodynamic expansion

$$v_i \sim \epsilon, \quad P \sim \epsilon^2, \quad \partial_i \sim \epsilon, \quad \partial_\tau \sim \epsilon^2, \quad \partial_r \sim \epsilon^0, \tag{76}$$

the metric which solves Einstein equations (4) up to order ϵ^4 is given as [3],

$$\begin{aligned}
\mathrm{d}s^2_{p+2} = & -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i - 2\big(1-\frac{r}{r_c}\big)v_i\mathrm{d}x^i\mathrm{d}\tau - \frac{2}{r_c}v_i\mathrm{d}x^i\mathrm{d}r \\
& + \big(1-\frac{r}{r_c}\big)\Big[(v^2+2P)\mathrm{d}\tau^2 + \frac{1}{r_c}v_iv_j\mathrm{d}x^i\mathrm{d}x^j\Big] + \frac{1}{r_c}\left(v^2+2P\right)\mathrm{d}\tau\mathrm{d}r \\
& + 2g^{(3)}_{\tau i}\mathrm{d}x^i\mathrm{d}\tau + g^{(4)}_{\tau\tau}\mathrm{d}\tau^2 + g^{(4)}_{ij}\mathrm{d}x^i\mathrm{d}x^j + O(\epsilon^5),
\end{aligned} \tag{77}$$

where

$$\begin{aligned}
g^{(3)}_{\tau i} = & \frac{(r-r_c)}{2r_c}\Big[(v^2+2P)\frac{2v_i}{r_c} + 4\partial_iP - (r+r_c)\partial^2v_i\Big], \\
g^{(4)}_{\tau\tau} = & -\frac{(r-r_c)^3}{2r_c^2}(\omega_{kl}\omega^{kl}) + \frac{(r-r_c)^2}{2r_c}(2v^k\partial^2v_k + \sigma_{kl}\sigma^{kl}) - \frac{(r-r_c)}{r_c}F^{(4)}_\tau, \\
F^{(4)}_\tau = & \frac{9}{8r_c}v^4 + \frac{5}{2r_c}Pv^2 + \frac{P^2}{r_c} - 2r_cv^k\partial^2v_k - 2r_c\sigma_{kl}\sigma^{kl} - 2\partial_\tau P + 2v^k\partial_kP, \\
g^{(4)}_{ij} = & (1-\frac{r}{r_c})\Big[\frac{1}{r_c^2}\left(v_iv_j - r_c\sigma_{ij}\right)\left(v^2+2P\right) + \frac{2}{r_c}v_{(i}\partial_{j)}P - \frac{1}{r_c}v_{(i}\partial_{j)}v^2 - \frac{r+r_c}{r_c}v_{(i}\partial^2v_{j)} \\
& - 2\sigma_{ik}\sigma^k{}_j - 4\sigma_{k(i}\omega^k{}_{j)} + \frac{r-5r_c}{r_c}\omega_{ik}\omega^k{}_j - 4\partial_i\partial_jP + \frac{r+5r_c}{2}\partial^2\sigma_{ij}\Big].
\end{aligned} \tag{78}$$

The dual fluid satisfies the incompressible Navier-Stokes equations with higher order corrections

$$\partial_iv^i = \theta, \qquad \partial_\tau v_i + v^j\partial_jv_i - r_c\partial^2v_i + \partial_iP = f_i, \tag{79}$$

where

$$\theta = -v^i\partial^2v_i + 2\sigma_{ij}\sigma^{ij} + \frac{1}{r_c}v^i\partial_iP + O(\epsilon^6), \tag{80}$$

$$\begin{aligned}
f_i = & -\frac{3r_c^2}{2}\partial^4v_i + 2r_cv^k\partial^2\partial_kv_i + 4r_c\sigma_{ik}\partial_l\sigma^{kl} - 10r_c\omega_{ik}\partial_l\sigma^{kl} - 3r_c\partial_i(\sigma_{kl}\sigma^{kl}) \\
& - \frac{5r_c}{2}\partial_i(\omega_{kl}\omega^{lk}) + 4r_c\sigma^{kl}\partial_k\sigma_{li} - 2v^k\partial_k\partial_iP - 2(\partial^kv_i)\partial_kP - (P+\frac{1}{2}v^2)\partial^2v_i \\
& - (\partial_k\sigma_{il})v^kv^l + (\partial_k\omega_{il})v^kv^l + 4(\partial_kv_i)\omega^{kl}v_l + r_c^{-1}(P+v^2)\partial_iP - r_c^{-1}v_i\partial_\tau P + O(\epsilon^7).
\end{aligned} \tag{81}$$

A.2 Near horizon expansion

An alternate presentation of the metric (77) was given in [2], through taking the coordinate transformations

$$\hat{x}^i = \epsilon\, r_c^{-1}x^i, \quad \hat{\tau} = \epsilon^2\, r_c^{-1}\tau, \quad \hat{r} = r_c^{-1}r, \tag{82}$$

so that $\hat{\partial}_i \equiv \frac{\partial}{\partial\hat{x}^i} \sim \epsilon^0$, $\partial_{\hat{\tau}} \sim \epsilon^0$, and $\partial_{\hat{r}} \sim \epsilon^0$. In the new coordinates one defines

$$\hat{P}(\hat{x},\hat{\tau}) = \epsilon^{-2}P(x(\hat{x}),\tau(\hat{\tau})), \quad \hat{v}_i(\hat{x},\hat{\tau}) = \epsilon^{-1}v_i(x(\hat{x}),\tau(\hat{\tau})), \tag{83}$$

and $\hat{v}^2 \equiv \hat{v}_i\delta^{ij}\hat{v}_j$. Considering the rescaled metric $\mathrm{d}\hat{s}^2_{p+2} = \epsilon^2\, r_c^{-2}\mathrm{d}s^2_{p+2}$ and defining $\lambda^2 = \epsilon^2\, r_c^{-1}$, one finds

$$\begin{aligned}
\mathrm{d}\hat{s}^2_{p+2} =& -\frac{\hat{r}}{\lambda^2}\mathrm{d}\hat{\tau}^2 + \left[2\mathrm{d}\hat{\tau}\mathrm{d}\hat{r} + \mathrm{d}\hat{x}_i\mathrm{d}\hat{x}^i - 2(1-\hat{r})\hat{v}_i\mathrm{d}\hat{x}^i\mathrm{d}\hat{\tau} + (1-\hat{r})(\hat{v}^2+2\hat{P})\mathrm{d}\hat{\tau}^2\right] \\
&+ \lambda^2\left[(1-\hat{r})\hat{v}_i\hat{v}_j\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j - 2\hat{v}_i\mathrm{d}\hat{x}^i\mathrm{d}\hat{r} + (\hat{v}^2+2\hat{P})\mathrm{d}\hat{\tau}\mathrm{d}\hat{r} + 2\hat{g}^{(2)}_{\hat{\tau}i}\mathrm{d}\hat{x}^i\mathrm{d}\hat{\tau} + \hat{g}^{(2)}_{\hat{\tau}\hat{\tau}}\mathrm{d}\hat{\tau}^2\right] \\
&+ \lambda^4\left[\hat{g}^{(4)}_{ij}\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j + 2\hat{g}^{(4)}_{\hat{\tau}i}\mathrm{d}\hat{x}^i\mathrm{d}\hat{\tau} + \hat{g}^{(4)}_{\hat{\tau}\hat{\tau}}\mathrm{d}\hat{\tau}^2\right] + O(\lambda^6),
\end{aligned} \tag{84}$$

where

$$\begin{aligned}
\hat{g}^{(2)}_{\hat{\tau}i} =& \frac{(\hat{r}-1)}{2}\Big[(\hat{v}^2+2\hat{P})2\hat{v}_i + 4\partial_i\hat{P} - (\hat{r}+1)\hat{\partial}^2 v_i\Big], \\
\hat{g}^{(2)}_{\hat{\tau}\hat{\tau}} =& -\frac{(\hat{r}-1)^3}{2}(\hat{\omega}_{kl}\hat{\omega}^{kl}) + \frac{(\hat{r}-1)^2}{2}(2\hat{v}^k\hat{\partial}^2\hat{v}_k + \hat{\sigma}_{kl}\hat{\sigma}^{kl}) - (\hat{r}-1)\hat{F}^{(2)}_{\hat{\tau}}, \\
\hat{F}^{(2)}_{\hat{\tau}} =& \frac{9}{8}\hat{v}^4 + \frac{5}{2}\hat{P}\hat{v}^2 + \hat{P}^2 - 2\hat{v}^k\hat{\partial}^2\hat{v}_k - 2\hat{\sigma}_{kl}\hat{\sigma}^{kl} - 2\hat{\partial}_{\hat{\tau}}\hat{P} + 2\hat{v}^k\hat{\partial}_k\hat{P}, \\
\hat{g}^{(4)}_{ij} =& (1-\hat{r})\Big[\left(\hat{v}_i\hat{v}_j - \hat{\sigma}_{ij}\right)\left(\hat{v}^2+2\hat{P}\right) + 2\hat{v}_{(i}\hat{\partial}_{j)}\hat{P} - \hat{v}_{(i}\hat{\partial}_{j)}\hat{v}^2 - (\hat{r}+1)\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} \\
&- 2\hat{\sigma}_{ik}\hat{\sigma}^k{}_j - 4\hat{\sigma}_{k(i}\hat{\omega}^k{}_{j)} + (\hat{r}-5)\hat{\omega}_{ik}\hat{\omega}^k{}_j - 4\hat{\partial}_i\hat{\partial}_j\hat{P} + \frac{\hat{r}+5}{2}\hat{\partial}^2\hat{\sigma}_{ij}\Big].
\end{aligned} \tag{85}$$

The incompressible Navier-Stokes equations (79) change into

$$\hat{\partial}_i\hat{v}^i = \hat{\theta}, \quad \partial_{\hat{\tau}}\hat{v}_i + \hat{v}^j\hat{\partial}_j\hat{v}_i - \hat{\partial}^2\hat{v}_i + \hat{\partial}_i\hat{P} = \hat{f}_i, \tag{86}$$

where

$$\hat{\theta} = \lambda^2\left[-\hat{v}^i\hat{\partial}^2\hat{v}_i + 2\hat{\sigma}_{ij}\hat{\sigma}^{ij} + \hat{v}^i\hat{\partial}_i\hat{P}\right] + O(\lambda^4), \tag{87}$$

$$\begin{aligned}
\hat{f}_i =& \lambda^2\Big[-\frac{3}{2}\hat{\partial}^4\hat{v}_i + 2\hat{v}^k\hat{\partial}^2\hat{\partial}_k\hat{v}_i + 4\hat{\sigma}_{ik}\hat{\partial}_l\hat{\sigma}^{kl} - 10\hat{\omega}_{ik}\hat{\partial}_l\hat{\sigma}^{kl} - 3\hat{\partial}_i(\hat{\sigma}_{kl}\hat{\sigma}^{kl}) - \frac{5}{2}\hat{\partial}_i(\hat{\omega}_{kl}\hat{\omega}^{lk}) \\
&+ 4\hat{\sigma}^{kl}\hat{\partial}_k\hat{\sigma}_{li} - 2\hat{v}^k\hat{\partial}_k\hat{\partial}_i\hat{P} - 2(\hat{\partial}^k\hat{v}_i)\hat{\partial}_k\hat{P} - (\hat{P} + \frac{1}{2}\hat{v}^2)\hat{\partial}^2\hat{v}_i - (\hat{\partial}_k\hat{\sigma}_{il})\hat{v}^k\hat{v}^l \\
&+ (\hat{\partial}_k\hat{\omega}_{il})\hat{v}^k\hat{v}^l + 4(\hat{\partial}_k\hat{v}_i)\hat{\omega}^{kl}\hat{v}_l + (\hat{P}+\hat{v}^2)\hat{\partial}_i\hat{P} - \hat{v}_i\partial_{\hat{\tau}}\hat{P}\Big] + O(\lambda^4)\,.
\end{aligned} \tag{88}$$

With these constraints the metric (84) solves the vacuum Einstein equations (4) up to order λ^0 consistently. To solve the next non-trivial order that at λ^2, especially the $\hat{\tau}\hat{\tau}$ and $\hat{\tau}i$ components, the terms $\hat{g}^{(4)}_{\hat{\tau}i}$ and $\hat{g}^{(4)}_{\hat{\tau}\hat{\tau}}$ are needed. We do not intend to find their explicit expressions here, as it is found that at order λ^2, they do not contribute to the Petrov type I equation in (2).

B The dual Fluid

To discuss the fluid dual to vacuum Einstein gravity, the theory of relativistic hydrodynamics up to second order in fluid gradients was presented in [3, 7, 15]. Choosing the

Landau frame of the relativistic fluid with velocity u^a so that its stress tensor is written as

$$T_{ab} = \mathrm{e}\, u_a u_b + \mathrm{p} h_{ab} + \Pi^{\perp}_{ab}, \qquad u^a \Pi^{\perp}_{ab} = 0, \tag{89}$$

where e and p represent the energy density and pressure of the fluid in the local rest frame. The induced metric $h_{ab} = \gamma_{ab} + u_a u_b$, with γ_{ab} the intrinsically flat metric and $\gamma_{ab} u^a u^b = -1$. The dissipative corrections can be written down through taking the isotropic gauge so that $\Pi^{\perp}_{ab}$ does not contain terms proportional to h_{ab}. Up to second order in gradients,

$$\begin{aligned}\Pi^{\perp}_{ab} = -2\eta \mathcal{K}_{ab} + \mathrm{p}^{-1} \big[& c_1 \mathcal{K}_{ca} \mathcal{K}^c{}_b + c_2 \mathcal{K}_{c(a} \Omega^c{}_{b)} + c_3 \Omega_{ac} \Omega^c{}_b + c_4 h^c_a h^d_b \partial_c \partial_d \ln \mathrm{p} \\ & + c_5 \mathcal{K}_{ab}\, D \ln \mathrm{p} + c_6 D^{\perp}_a \ln \mathrm{p}\, D^{\perp}_b \ln \mathrm{p} \big],\end{aligned} \tag{90}$$

where $D^{\perp}_a \equiv h_a^b \partial_b$, $D \equiv u^a \partial_a$ have been defined, η is the relativistic kinematic shear viscosity, and $c_1, ..., c_6$ are the corresponding transport coefficients at the second order. The equations of motion $\partial^b T_{ab}$ at the lowest order have been considered in writing down the above form, and the relativistic shear and vorticity are defined as

$$\mathcal{K}_{ab} = h^c_a h^d_b \partial_{(c} u_{d)}, \qquad \Omega_{ab} = h^c_a h^d_b \partial_{[c} u_{d]}. \tag{91}$$

The energy density which vanishes for equilibrium configurations can also be expanded as

$$\mathrm{e} = \zeta' D \ln p + \mathrm{p}^{-1} \left[d_1 \mathcal{K}_{ab} \mathcal{K}^{ab} + d_2 \Omega_{ab} \Omega^{ab} + d_3 (D \ln \mathrm{p})^2 + d_4 D D \ln \mathrm{p} + d_5 (D_{\perp} \ln \mathrm{p})^2 \right], \tag{92}$$

where ζ' is an alternative first order transport coefficient which is similar to the bulk viscosity that measures variations of the energy density, and $d_1, ..., d_5$ are the corresponding second order transport coefficients. However, these six coefficients are not independent [15], if we consider the equation of state of this special fluid dual to vacuum Einstein gravity that $T^2 - pT_{ab}T^{ab} = 0$, which comes from the Hamiltonian constraint (9). Taking account of the expansions (89) and (90), one finds the energy density e can be expressed as

$$\mathrm{e} = -2\eta^2 \mathrm{p}^{-1} \mathcal{K}_{ab} \mathcal{K}^{ab} + O(\partial^3). \tag{93}$$

Comparing (92) with (93), one can read off

$$\zeta' = 0, \quad d_1 = -2\eta^2, \quad d_2 = d_3 = d_4 = d_5 = 0, \tag{94}$$

Thus, in this paper we only consider the independent transport coefficients in (90).

B.1 Non-relativistic hydrodynamic expansion

With the pressure $\mathrm{p} = r_c{}^{1/2} + r_c^{3/2} P$, the full fluid stress tensor (89) can be expanded up to order ϵ^4 through the non-relativistic hydrodynamical expansion (15) as

$$T^{\tau}{}_i = + r_c^{-3/2} v_i + r_c^{-5/2} \left[v_i(v^2 + P) - 2\eta r_c \sigma_{ij} v^j \right] + O(\epsilon^5), \tag{95}$$

$$T^{\tau}{}_{\tau} = - r_c^{-3/2} v^2 - r_c^{-5/2} \left[v^2(v^2 + P) - 2\eta r_c \sigma_{ij} v^i v^j - 2\eta^2 r_c^2 \sigma_{ij} \sigma^{ij} \right] + O(\epsilon^6), \tag{96}$$

$$\begin{aligned} T_{ij} = &+ r_c^{-1/2} \delta_{ij} + r_c^{-3/2} \left[P\delta_{ij} + v_i v_j - 2\eta r_c \partial_{(i} v_{j)} \right] \\ &+ r_c^{-5/2} \left[v_i v_j (v^2 + P) - \eta r_c \sigma_{ij} v^2 + 2\eta r_c v_{(i}\partial_{j)} P - \eta r_c v_{(i}\partial_{j)} v^2 - 2\eta^2 r_c^2 v_{(i}\partial^2 v_{j)} \right. \\ &\left. + c_1\, r_c^2 \sigma_{ik}\sigma^k{}_j + c_2\, r_c^2 \sigma_{k(i}\omega^k{}_{j)} + c_3\, r_c^2 \omega_{ik}\omega^k{}_j + c_4\, r_c^2 \partial_i\partial_j P \right] + O(\epsilon^6), \end{aligned} \tag{97}$$

$$T \;= T^{\tau}{}_{\tau} + T^i{}_i = p\, r_c^{-1/2} + p\, r_c^{-3/2} P + O(\epsilon^6), \tag{98}$$

where the equations of motion $\partial^b T_{ab} = 0$ at lower orders have been employed.

B.2 Alternate presentation

With the coordinates in (82), considering the re-scaled stress tensor

$$\hat{T}_{ab} \mathrm{d}\hat{x}^a \mathrm{d}\hat{x}^b = r_c^{-1} \epsilon\, T_{ab} \mathrm{d}x^a \mathrm{d}x^b, \quad \lambda^2 \equiv r_c^{-1} \epsilon^2, \tag{99}$$

one finds the stress tensor (95)-(98) can be transformed into

$$\hat{T}^{\hat{\tau}}{}_i = + \lambda v_i + \lambda^3 \left[\hat{v}_i(\hat{v}^2 + \hat{P}) - 2\hat{\eta}\hat{\sigma}_{ij}\hat{v}^j \right] + O(\lambda^5), \tag{100}$$

$$\hat{T}^{\hat{\tau}}{}_{\hat{\tau}} = - \lambda v^2 - \lambda^3 \left[\hat{v}^2(\hat{v}^2 + \hat{P}) - 2\hat{\eta}\hat{\sigma}_{ij}\hat{v}^i\hat{v}^j - 2\hat{\eta}^2 \hat{\sigma}_{ij}\hat{\sigma}^{ij} \right] + O(\lambda^5), \tag{101}$$

$$\begin{aligned} \hat{T}_{ij} = &+ \lambda^{-1} \delta_{ij} + \lambda \left[\hat{P}\delta_{ij} + \hat{v}_i\hat{v}_j - 2\hat{\eta}\hat{\sigma}_{ij} \right] \\ &+ \lambda^3 \left[\hat{v}_i\hat{v}_j(\hat{v}^2 + \hat{P}) - \hat{\eta}\hat{\sigma}_{ij}\hat{v}^2 + 2\hat{\eta}\hat{v}_{(i}\hat{\partial}_{j)}\hat{P} - \hat{\eta}\hat{v}_{(i}\hat{\partial}_{j)}\hat{v}^2 - 2\hat{\eta}^2\hat{v}_{(i}\hat{\partial}^2\hat{v}_{j)} \right. \\ &\left. + \hat{c}_1\, \hat{\sigma}_{ik}\hat{\sigma}^k{}_j + \hat{c}_2\, \hat{\sigma}_{k(i}\hat{\omega}^k{}_{j)} + \hat{c}_3\, \hat{\omega}_{ik}\hat{\omega}^k{}_j + \hat{c}_4\, \hat{\partial}_i\hat{\partial}_j\hat{P} \right] + O(\lambda^5), \end{aligned} \tag{102}$$

$$\hat{T} \;= \hat{T}^{\hat{\tau}}{}_{\hat{\tau}} + \hat{T}^i{}_i = \lambda^{-1} p + \lambda\, p\, \hat{P} + O(\lambda^5). \tag{103}$$

This is also the Brown-York stress tensor dual to the metric (84), which is mathematically equivalent to the metric with the near horizon expansion [2].

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